

A 1D Model for N-Level Atoms Coupled to an EM Field

Zorawar S. Bassi
André LeClair

Newman Laboratory
Cornell University
Ithaca, NY 14853

We construct a model for n-level atoms coupled to quantized electromagnetic fields in a fibrillar geometry. In the slowly varying envelope and rotating wave approximations, the equations of motion are shown to satisfy a zero curvature representation, implying integrability of the quantum system.

Introduction

The interaction of radiation with two-level atoms has been extensively studied under various approximations. In one spacial dimension, the reduced Maxwell-Bloch equations resulting from the slowly varying envelope and rotating wave approximations are known to be quantum integrable [1]. In this paper we generalize the one-dimensional case of two-level atoms to that of n-level atoms.

In the first of two parts, we construct the fully quantum n-level model. The system consists of n-level atoms distributed in a fibrillar geometry, interacting with radiation through a minimally coupled hamiltonian. In the remaining section we apply the approximations, and show that the Heisenberg equations of motion for the reduced system satisfy the so-called zero curvature representation. This implies that the system is integrable and can be solved by the inverse scattering method.

0. Mathematical Background

Let us first recall the sl_n Lie algebra. The $n^2 - 1$ generators, written as

$$\{E_{ij}, H_m \mid 1 \leq i \neq j \leq n, 1 \leq m \leq r\}, \quad (0.1)$$

where $r = n - 1$ is the rank, satisfy the following brackets (in the Chevalley basis)

$$[E_{ij}, E_{kl}] = \begin{cases} \delta_{kj}E_{il} - \delta_{il}E_{kj}, & \text{if } \delta_{il}\delta_{kj} = 0 \\ \sum_{m=i}^{j-1} H_m (i < j), & \text{if } \delta_{il}\delta_{kj} = 1 \end{cases}, \quad (0.2a)$$

$$[E_{ij}, H_m] = (\delta_{jm} - \delta_{im} - \delta_{jm+1} + \delta_{im+1})E_{ij}, \quad (0.2b)$$

$$[H_a, H_b] = 0. \quad (0.2c)$$

The set spanned by $\{H_m\}$ being the Cartan subalgebra. (Note that in the bracket (0.2a) a term of the form $\delta_{i \neq j}E_{kk}$ is formally equal to zero, even though E_{kk} has not been defined. The set $\{E_{ij} \mid 1 \leq i, j \leq n\}$ satisfying the first relation in (0.2a) is a basis for the algebra gl_n .) A representation ρ of sl_n will be denoted as

$$\{E_{ij}^\rho = \rho(E_{ij}), H_m^\rho = \rho(H_m)\}. \quad (0.3)$$

The $r \times r$ Cartan matrix A has the explicit form

$$A_{uv} = 2\delta_{uv} - \delta_{uv-1} - \delta_{uv+1}. \quad (0.4)$$

It is a symmetric matrix with diagonal elements 2 and nearest off-diagonal elements -1 . We now proceed to build our quantum system.

1. The Interacting N-Level Hamiltonian

We model a free n-level atom as having a single electron with eigenstates $|i\rangle$, $i = 1, 2, \dots, n$, and energies $\epsilon_1 > \epsilon_2 > \dots > \epsilon_n$. The energy splitting between states $|i\rangle$ and $|i+1\rangle$ will be denoted by ω_i or ω_{ii+1} ($\hbar = 1$)

$$\omega_i = \omega_{ii+1} = \epsilon_i - \epsilon_{i+1}, \quad 1 \leq i < n. \quad (1.1)$$

The second notation can be generalized as follows

$$\omega_{ij} = \epsilon_i - \epsilon_j, \quad 1 \leq i < j \leq n. \quad (1.2)$$

The notation ω_i is only defined for the energy splitting between successive states.

To define various atomic operators, we first introduce fermion creation and destruction operators $\{b_i, b_i^\dagger\}$ for $1 \leq i \leq n$. The operator b_i^\dagger (b_i) creates (destroys) an electron in the i -th level. These operators satisfy the algebra

$$\{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0, \quad \{b_i, b_j^\dagger\} = \delta_{ij}. \quad (1.3)$$

The atomic operators can now be written as

$$\mathcal{O}_{ij} = b_i^\dagger b_j, \quad 1 \leq i, j \leq n \quad (1.4)$$

or linear combinations of the \mathcal{O}_{ij} s. The action of \mathcal{O}_{ij} on an atomic state $|k\rangle$ is given by

$$\mathcal{O}_{ij}|k\rangle = b_i^\dagger b_j |k\rangle = \delta_{kj} |i\rangle. \quad (1.5)$$

From (1.3) the general commutator for the \mathcal{O} operators is

$$[\mathcal{O}_{ij}, \mathcal{O}_{kl}] = \mathcal{O}_{il}\delta_{jk} - \mathcal{O}_{kj}\delta_{il}. \quad (1.6)$$

Operators of the form $\mathcal{O}_{i < j}$ are referred to as raising operators. These cause a transition from the lower energy state $|j\rangle$ to the higher energy state $|i\rangle$. Similarly the operators $\mathcal{O}_{j > i}$ are lowering operators. We also define a set of commuting operators, denoted \mathcal{H}_m , as follows

$$\mathcal{H}_m = \mathcal{O}_{mm} - \mathcal{O}_{m+1m+1}, \quad 1 \leq m \leq r. \quad (1.7)$$

The set $\{\mathcal{O}_{i \neq j}, \mathcal{H}_m\}$ satisfies (0.2), thus forming a representation of sl_n . (Note: We shall often use the notation $X_{a < b}$ ($X_{a > b}$) to mean X_{ab} with $a < b$ ($a > b$) for X any quantity, operator, c-number, etc.)

By appropriately choosing the arbitrary lowest state energy ϵ_n to be

$$\epsilon_n = - \sum_{m=1}^r A_{rm}^{-1} \omega_m, \quad (1.8)$$

where A^{-1} is the inverse Cartan matrix, the free atomic hamiltonian can be written as

$$H_0^{\text{atom}} = \sum_{1 \leq u, v \leq r} A_{uv}^{-1} \omega_v \mathcal{H}_u. \quad (1.9)$$

To couple the atom to an electromagnetic field we make use of the minimal coupling prescription. The standard hamiltonian is

$$H = H_0^\phi + H_0^{\text{atom}} + H_{\text{int}}, \quad (1.10)$$

where

$$H_0^{\text{atom}} = \frac{1}{2m_e} \vec{p} \cdot \vec{p} + V(\vec{x}) \quad (1.11)$$

$$H_{\text{int}} = -\frac{e}{2m_e} (\vec{p} \cdot \vec{A}(\vec{x}) + \vec{A}(\vec{x}) \cdot \vec{p}) + \frac{e^2}{2m} \vec{A}(\vec{x}) \cdot \vec{A}(\vec{x}), \quad (1.12)$$

and H_0^ϕ is the free field hamiltonian. If the spatial variation of the vector potential \vec{A} is small across the atom, we can take its value at a fixed point \vec{x}_0 inside the atom. Using

$$\vec{p} = -i \frac{m_e}{\hbar} [\vec{x}, H_0^{\text{atom}}], \quad (1.13)$$

we get

$$-\frac{e}{2m_e} \langle a | \vec{p} \cdot \vec{A}(\vec{x}) + \vec{A}(\vec{x}) \cdot \vec{p} | b \rangle = \frac{i}{\hbar} (\epsilon_b - \epsilon_a) \vec{A}(\vec{x}_0) \cdot \langle a | \vec{d} | b \rangle, \quad (1.14)$$

where $\vec{d} = e\vec{x}$ is the electric dipole operator. Since \vec{d} is a vector operator and the atomic states are assumed to be parity eigenstates, we have $\langle i | \vec{d} | i \rangle = 0$. For $i < j$, the matrix elements are of the form

$$\langle i | \vec{d} | j \rangle = d_{ij} e^{i\alpha_{ij}} \hat{n}, \quad \langle j | \vec{d} | i \rangle = d_{ij} e^{-i\alpha_{ij}} \hat{n}, \quad (1.15)$$

where $d_{ij} \geq 0$ and the unit vector \hat{n} gives the spatial orientation of the atom. This shows that the dipole operator can be expanded in terms of the raising and lowering operators as

$$\vec{d} = \hat{n} \sum_{i < j} (d_{ij} e^{i\alpha_{ij}} \mathcal{O}_{ij} + d_{ij} e^{-i\alpha_{ij}} \mathcal{O}_{ji}). \quad (1.16)$$

The interaction hamiltonian becomes

$$H_{\text{int}} = -i\vec{A}(\vec{x}_0) \cdot \hat{n} \sum_{i < j} \omega_{ij} d_{ij} (\mathcal{O}_{ij} e^{i\alpha_{ij}} - \mathcal{O}_{ji} e^{-i\alpha_{ij}}) + \frac{e^2}{2m_e} \vec{A}(\vec{x}_0) \cdot \vec{A}(\vec{x}_0). \quad (1.17)$$

To reduce this system to a one-dimensional model, we make use of the fibrillar geometry. The atom can be thought of as an impurity in an optical fiber of cross-sectional area \mathcal{A} and length L , with $L \gg \sqrt{\mathcal{A}}$. Taking the fiber along the \hat{x} direction, the reduced field action is found to be (see [2] for details)

$$\frac{1}{\hbar} S_{\text{Maxwell}} = \int dx dt \frac{1}{2} (\partial_t \phi \partial_t \phi - \partial_x \phi \partial_x \phi), \quad (1.18)$$

where ϕ is a dimensionless scalar field defined through

$$\vec{A} \cdot \hat{n} = \sqrt{\frac{4\pi\hbar}{\mathcal{A}_{\text{eff}}}} \phi. \quad (1.19)$$

Here \vec{A} is the vector potential depending only on the x coordinate and \mathcal{A}_{eff} is the effective fiber cross-sectional area. The field ϕ satisfies the commutation relation

$$[\phi(x, t), \partial_t \phi(x', t)] = i\delta(x - x'). \quad (1.20)$$

From the action the free field hamiltonian is found to be

$$H_0^\phi = \int dx \frac{1}{2} [(\partial_t \phi)^2 + (\partial_x \phi)^2] + \frac{2\pi e^2}{m_e \mathcal{A}_{\text{eff}}} \phi^2(x_0), \quad (1.21)$$

where the last term is the quadratic potential term taken from H_{int} . Now the interaction Hamiltonian is

$$H_{\text{int}} = -\frac{i}{2} \phi(x_0) \sum_{i < j} \omega_{ij} \beta_{ij} (\mathcal{O}_{ij} e^{i\alpha_{ij}} - \mathcal{O}_{ji} e^{-i\alpha_{ij}}), \quad (1.22)$$

where (explicitly showing \hbar and c)

$$\beta_{i < j} = \sqrt{\frac{16\pi}{\hbar c \mathcal{A}_{\text{eff}}}} d_{ij}. \quad (1.23a)$$

For $1 \leq m \leq r$ we define

$$\beta_m = \sqrt{\frac{16\pi}{\hbar c \mathcal{A}_{\text{eff}}}} d_{mm+1} = \beta_{mm+1}. \quad (1.23b)$$

The β parameters are the important dimensionless coupling constants of the model. The spontaneous decay rate $\Gamma_{ij}^s = 1/\tau_{ij}^s$ of a single excited atom from the state $|i\rangle$ to the state $|j\rangle$ is given by

$$\Gamma_{ij}^s = \frac{\beta_{ij}^2}{4} \omega_{ij}. \quad (1.24)$$

Next to make the transition to a continuous system. For N atoms positioned at $x = x_m, m = 1, \dots, N$, let $\vec{d}_m = e(\vec{x} - \vec{x}_m)$ and $\mathcal{O}_{ij}(x_m)$ be the dipole and transition operators on the atom at x_m . The matrix

elements of \vec{d}_m are independent of the position, however, the orientation can vary from atom to atom. The operator \vec{d}_m can be written in terms of the single atom matrix elements as

$$\vec{d}_m = \hat{n}_m \sum_{i < j} d_{ij} (\mathcal{O}_{ij}(x_m) e^{i\alpha_{ij}} + \mathcal{O}_{ji}(x_m) e^{-i\alpha_{ij}}). \quad (1.25)$$

For simplification, we consider the situation where all atoms are aligned $\hat{n}_m = \hat{n}$ (e.g., by an external electric field), giving

$$H_{\text{int}} = -\frac{i}{2} \int dx \phi(x) \sum_{i < j} \omega_{ij} \beta_{ij} (\mathcal{O}_{ij}(x, t) e^{i\alpha_{ij}} - \mathcal{O}_{ji}(x, t) e^{-i\alpha_{ij}}), \quad (1.26)$$

where we have introduced the space-time dependent transition operators

$$\mathcal{O}_{ij}(x, t) = \sum_{m=1}^N \mathcal{O}_{ij}(x_m, t) \delta(x - x_m). \quad (1.27)$$

The discrete operator $\mathcal{O}_{ij}(x_m, t)$ acts only on the atom at x_m to cause a transition from $|j\rangle$ to $|i\rangle$. The $\mathcal{H}_m(x, t)$ operators are defined similarly. The general commutator for the space-time transition operators is

$$[\mathcal{O}_{ij}(x, t), \mathcal{O}_{kl}(x', t)] = (\mathcal{O}_{il}(x, t) \delta_{jk} - \mathcal{O}_{kj}(x, t) \delta_{il}) \delta(x - x'), \quad (1.28)$$

from which it is easily seen that the algebra (or now more appropriately current algebra) satisfied by the set $\{\mathcal{O}_{i \neq j}(x, t), \mathcal{H}_m(x, t)\}$ is identical, aside from the delta function factor, to the sl_n algebra (0.2). The free atomic hamiltonian takes the form

$$H_0^{\text{atom}} = \int dx \sum_{1 \leq u, v \leq r} A_{uv}^{-1} \omega_v \mathcal{H}_u(x, t). \quad (1.29)$$

The complete hamiltonian for the system is therefore $H_0^{\text{atom}} + H_0^\phi + H_{\text{int}}$, with H_0^{atom} , H_0^ϕ and H_{int} given by (1.29), (1.21) and (1.26) respectively.

2. Two Approximations and Integrability

We now make use of two approximations common in quantum optics to further simplify H_{int} . These being the slowly varying envelope and rotating wave approximations.

In the slowly varying envelope approximation, we assume that near resonant photons with energies $\approx \omega_{ij}$ are most relevant. Then the scalar field ϕ can be expanded about the various resonances as

$$\phi(x, t) \approx \sum_{i < j} \left(e^{-i\omega_{ij}(t-x)} \psi_{ij}(x, t) + e^{i\omega_{ij}(t-x)} \psi_{ij}^\dagger(x, t) \right), \quad (2.1)$$

where $\psi_{ij}(x, t)$ and $\psi_{ij}^\dagger(x, t)$ are destruction and creation fields with mode expansions

$$\psi_{i < j}(x, 0) = \frac{1}{\sqrt{2\omega_{ij}}} \int \frac{dk_e}{\sqrt{2\pi}} \hat{a}_{ij}(k_e) e^{ik_e x}, \quad (2.2a)$$

$$\psi_{i < j}^\dagger(x, 0) = \frac{1}{\sqrt{2\omega_{ij}}} \int \frac{dk_e}{\sqrt{2\pi}} \hat{a}_{ij}^\dagger(k_e) e^{-ik_e x}, \quad (2.2b)$$

and

$$\hat{a}_{ij}(k_e) = a_{ij}(k_e + \omega_{ij}), \quad \hat{a}_{ij}^\dagger(k_e) = a_{ij}^\dagger(k_e + \omega_{ij}). \quad (2.3)$$

Here a_{ij} (a_{ij}^\dagger) is the usual photon destruction (creation) operator. (Note that we are only considering right-moving plane waves.) The operator $\hat{a}_{ij}(k_e)$ ($\hat{a}_{ij}^\dagger(k_e)$) destroys (creates) a photon with energy

$$|k| = |k_e + \omega_{ij}| \approx \omega_{ij}. \quad (2.4)$$

Thus k_e acts as an ‘‘envelope’’ vector about the ω_{ij} resonance. The photon operators satisfy the standard commutator

$$[\hat{a}_{ij}(k), \hat{a}_{kl}^\dagger(k')] = [a_{ij}(k), a_{kl}^\dagger(k')] = \delta_{ik}\delta_{jl}\delta(k - k'). \quad (2.5)$$

All other commutators vanish. In writing (2.1) and the above commutaion relations, we have assumed that all resonances ω_{ij} ($\frac{1}{2}(n^2 - n)$ in total) are distinct (i.e., $\omega_{ij} = \omega_{kl} \iff i = l$ and $j = k$) and sharp. From (2.5) the component fields satisfy

$$[\psi_{ij}(x, t), \psi_{kl}^\dagger(x', t)] = \frac{1}{2\omega_{ij}}\delta_{ik}\delta_{jl}\delta(x - x'), \quad (2.6)$$

with all other commutators zero.

The rotating wave approximation reduces the number of interactions in H_{int} . Using (2.1) in H_{int} we obtain terms with both photon creation (a^\dagger) and atomic raising ($\mathcal{O}_{i < j}$) operators (or photon destruction and atomic lowering $\{a, \mathcal{O}_{j > i}\}$), or two photon creation (destruction) operator terms in H_0^ϕ . Such high frequency terms lead to vacuum fluctuations and higher order processes. The rotating wave approximation sets these processes to zero. We also set to zero terms of the form $\psi_{i < j}\mathcal{O}_{k < l}$ (or $\psi_{i < j}^\dagger\mathcal{O}_{l > k}$) for $(i, j) \neq (k, l)$ since they give no contribution to lowest order in perturbation theory. So we only retain those terms which pair creation/lowering or destruction/raising operators (and creation/destruction operators in H_0^ϕ) and connect states with \approx equal energy.

Combining these two approximations we get

$$H_{\text{int}} = -\frac{i}{2} \int dx \sum_{i < j} \beta_{ij}\omega_{ij} (\psi_{ij}e^{i\alpha_{ij}}\mathcal{O}_{ij}e^{-i\omega_{ij}(t-x)} - \psi_{ij}^\dagger e^{-i\alpha_{ij}}\mathcal{O}_{ji}e^{i\omega_{ij}(t-x)}), \quad (2.7)$$

$$H_0^\phi = -2i \int dx \sum_{i < j} \omega_{ij}\psi_{ij}^\dagger \partial_x \psi_{ij}. \quad (2.8)$$

The free field hamiltonian follows from the field action, which using

$$|k_e|^2 \ll \omega_{ij}^2 \implies |\partial_x \psi_{ij}| \ll \omega_{ij}|\psi_{ij}|, \quad |\partial_t \psi_{ij}| \ll \omega_{ij}|\psi_{ij}|, \quad (2.9)$$

approximates to

$$\int dx dt \frac{1}{2}((\partial_t \phi)^2 - (\partial_x \phi)^2) \approx 2i \int dx dt \sum_{i < j} \omega_{ij}\psi_{ij}^\dagger(\partial_x + \partial_t)\psi_{ij}. \quad (2.10)$$

(In (2.8) we have dropped the quadratic term that appears in (1.21). For electric fields small compared with e/a_0^2 , this term is negligible in relation to H_{int} .) The phases $e^{\pm i\alpha_{ij}}$ and $e^{\pm i\omega_{ij}x}$ can be absorbed into $\{\psi_{ij}, \psi_{ij}^\dagger\}$ and $\{\mathcal{O}_{ij}, \mathcal{O}_{ji}\}_{i < j}$ respectively without changing the commutation relations. The time dependent phase $e^{-i\omega_{ij}t}$ ($e^{+i\omega_{ij}t}$) cancels the time dependence of $\mathcal{O}_{i < j}$ ($\mathcal{O}_{j > i}$) coming from the free atomic hamiltonian. Thus we can set H_0^{atom} to zero and consider the model defined by the complete hamiltonian

$$H = -2i \int dx \sum_{i < j} \omega_{ij}\psi_{ij}^\dagger \partial_x \psi_{ij} - \frac{i}{2} \int dx \sum_{i < j} \omega_{ij}\beta_{ij}(\psi_{ij}\mathcal{O}_{ij} - \psi_{ij}^\dagger\mathcal{O}_{ji}). \quad (2.11)$$

Finally we can rescale ψ_{ij} and ψ_{ij}^\dagger as

$$\psi_{ij} \longrightarrow \frac{\psi_{ij}}{\sqrt{2\omega_{ij}}}, \quad \psi_{ij}^\dagger \longrightarrow \frac{\psi_{ij}^\dagger}{\sqrt{2\omega_{ij}}}, \quad (2.12)$$

which gives for the commutator (2.6)

$$[\psi_{ij}(x, t), \psi_{kl}^\dagger(x', t)] = \delta_{ik}\delta_{jl}\delta(x - x'), \quad (2.13)$$

and defining

$$\tilde{\omega}_{i < j} = \frac{1}{2\sqrt{2}} \beta_{ij} \sqrt{\omega_{ij}} = \sqrt{\frac{\Gamma_{ij}^s}{2}}, \quad \tilde{\omega}_m = \tilde{\omega}_{mm+1} \ (1 \leq m \leq r), \quad (2.14)$$

the hamiltonian (2.11) becomes

$$H = -i \int dx \sum_{i < j} \psi_{ij}^\dagger \partial_x \psi_{ij} - i \int dx \sum_{i < j} \tilde{\omega}_{ij} (\psi_{ij} \mathcal{O}_{ij} - \psi_{ij}^\dagger \mathcal{O}_{ji}). \quad (2.15)$$

The first interaction term, $\psi_{ij} \mathcal{O}_{ij}$, causes an atomic transition from a lower energy state $|j\rangle$ to a higher energy state $|i\rangle$, along with the absorption of a photon of energy $\approx \omega_{ij}$. The second term, $\psi_{ij}^\dagger \mathcal{O}_{ji}$, causes a transition from a higher energy state $|i\rangle$ to a lower energy state $|j\rangle$, along with the creation of a photon of energy $\approx \omega_{ij}$.

From H we can obtain the Heisenberg operator equations of motion. Explicitly we find

$$\begin{aligned} \partial_t \mathcal{O}_{k < l} = & - \sum_{j > l} \tilde{\omega}_{lj} \psi_{lj} \mathcal{O}_{kj} + \sum_{i < k} \tilde{\omega}_{ik} \psi_{ik} \mathcal{O}_{il} + \sum_{\substack{i < l \\ i \neq k}} \tilde{\omega}_{il} \psi_{il}^\dagger \mathcal{O}_{ki} \\ & - \sum_{\substack{j > k \\ j \neq l}} \tilde{\omega}_{kj} \psi_{kj}^\dagger \mathcal{O}_{jl} + \tilde{\omega}_{kl} \psi_{kl}^\dagger \sum_{m=k}^{l-1} \mathcal{H}_m, \end{aligned} \quad (2.16a)$$

$$\begin{aligned} \partial_t \mathcal{O}_{l > k} = \partial_t \mathcal{O}_{k < l}^\dagger = & - \sum_{j > l} \tilde{\omega}_{lj} \psi_{lj}^\dagger \mathcal{O}_{jk} + \sum_{i < k} \tilde{\omega}_{ik} \psi_{ik}^\dagger \mathcal{O}_{li} + \sum_{\substack{i < l \\ i \neq k}} \tilde{\omega}_{il} \psi_{il} \mathcal{O}_{ik} \\ & - \sum_{\substack{j > k \\ j \neq l}} \tilde{\omega}_{kj} \psi_{kj} \mathcal{O}_{lj} + \tilde{\omega}_{kl} \psi_{kl} \sum_{m=k}^{l-1} \mathcal{H}_m, \end{aligned} \quad (2.16b)$$

$$\begin{aligned} \partial_t \mathcal{H}_m = & -2\tilde{\omega}_{mm+1} (\psi_{mm+1} \mathcal{O}_{mm+1} + \psi_{mm+1}^\dagger \mathcal{O}_{m+1m}) - \sum_{j > m+1} \tilde{\omega}_{mj} (\psi_{mj} \mathcal{O}_{mj} + \psi_{mj}^\dagger \mathcal{O}_{jm}) \\ & - \sum_{i < m} \tilde{\omega}_{im+1} (\psi_{im+1} \mathcal{O}_{im+1} + \psi_{im+1}^\dagger \mathcal{O}_{m+1i}) + \sum_{i < m} \tilde{\omega}_{im} (\psi_{im} \mathcal{O}_{im} + \psi_{im}^\dagger \mathcal{O}_{mi}) \\ & + \sum_{j > m+1} \tilde{\omega}_{m+1j} (\psi_{m+1j} \mathcal{O}_{m+1j} + \psi_{m+1j}^\dagger \mathcal{O}_{jm+1}) \end{aligned} \quad (2.16c)$$

$$(\partial_t + \partial_x) \psi_{k < l} = \tilde{\omega}_{kl} \mathcal{O}_{lk} \quad (2.16d)$$

$$(\partial_t + \partial_x) \psi_{k < l}^\dagger = \tilde{\omega}_{kl} \mathcal{O}_{kl}. \quad (2.16e)$$

Each term in the equations of motion for the atomic operators has a simple physical interpretation. For example, consider the raising operator $\mathcal{O}_{k < l}$. The first summation in (2.16a) is a sum over all atomic and photon operator pairs, where the photon field destroys a photon of energy $\omega_{l < j}$, and the atomic (raising) operator causes a transition from the lower state $|j\rangle$ to the higher state $|k\rangle$. The change in energy of the system corresponding to the atomic/field pair being ω_{kl} (as is for every other term in (2.16a)). If we think of a field destroying (creating) a photon of energy $\omega_{i < j}$ as “connecting” atomic states $|i\rangle$ to $|j\rangle$ ($|j\rangle$ to $|i\rangle$), along with $\mathcal{O}_{i < j}$ ($\mathcal{O}_{j > i}$) connecting $|j\rangle$ to $|i\rangle$ ($|i\rangle$ to $|j\rangle$) and \mathcal{H}_m connecting $|m\rangle$ to $|m\rangle$ and $|m+1\rangle$ to $|m+1\rangle$, then (2.16a) is, aside from c-number factors, a sum over all atomic/field pairs connecting $|l\rangle$ to $|k\rangle$ through some intermediate state, i.e., $|l\rangle \rightarrow |j\rangle \rightarrow |k\rangle$. A similar interpretation follows for $\mathcal{O}_{l > k}$ and \mathcal{H}_m .

These equations of motion have a zero-curvature representation

$$[\partial_t + A_t, \partial_x + A_x] = 0, \quad (2.17a)$$

$$\implies \partial_t A_x - \partial_x A_t = [A_x, A_t], \quad (2.17b)$$

where A_x and A_t are matrices of quantum operators given by

$$A_x = \mu \sum_{1 \leq m, n \leq r} A_{mn}^{-1} \mathcal{H}_n H_m^\rho + \mu \sum_{i < j} (\mathcal{O}_{ij} E_{ij}^\rho + \mathcal{O}_{ji} E_{ji}^\rho) + \sum_{i < j} \tilde{\omega}_{ij} (-\psi_{ij}^\dagger E_{ij}^\rho + \psi_{ij} E_{ji}^\rho) - \frac{1}{\mu} \sum_{1 \leq m, n \leq r} A_{mn}^{-1} \tilde{\omega}_n^2 H_m^\rho \quad (2.18a)$$

$$A_t = \frac{1}{\mu} \sum_{1 \leq m, n \leq r} A_{mn}^{-1} \tilde{\omega}_n^2 H_m^\rho - \sum_{i < j} \tilde{\omega}_{ij} (-\psi_{ij}^\dagger E_{ij}^\rho + \psi_{ij} E_{ji}^\rho), \quad (2.18b)$$

provided that the $\tilde{\omega}_{i < j}$ s satisfy

$$\tilde{\omega}_{i < j}^2 = \tilde{\omega}_{i < k}^2 + \tilde{\omega}_{k < j}^2, \quad (2.19)$$

for any intermediate value of k , $i < k < j$. This is trivially satisfied by $\tilde{\omega}_{ii+1}$, in which case there is no such k . Here A_{mn}^{-1} is the inverse Cartan matrix, and $\{E_{i \neq j}^\rho, H_m^\rho\}$ are matrices in any representation ρ of sl_n satisfying (0.2). Requiring (2.17) to be valid for all values of the arbitrary spectral parameter μ , is equivalent to the equations of motion (2.16) (of course provided (2.19) is satisfied). The equivalence can be shown by making use of the commutation relations (0.2). From (2.16c) we can derive a more compact form for the equations of motion corresponding to the \mathcal{H}_m basis operators

$$\partial_t \mathcal{H}_m = - \sum_{u=1}^r A_{mu} \sum_{u < l \leq n} \sum_{k=1}^u \tilde{\omega}_{kl} (\mathcal{O}_{kl} \psi_{kl} + \mathcal{O}_{lk} \psi_{kl}^\dagger). \quad (2.20)$$

The constraint (2.19) arises in forming a zero-curvature representation for the field $(\{\psi_{ij}, \psi_{ij}^\dagger\})$ equations of motion, and reduces the number of free parameters to $r = n - 1$, these being $\{\tilde{\omega}_m\}_{1 \leq m \leq r}$. The definition (2.14) shows that the constraint is equivalent to the requirement that the spontaneous decay rate (for a single atom) from $|i\rangle$ to $|j\rangle$, Γ_{ij}^s , be equal to the sum of the decay rates Γ_{ik}^s and Γ_{kj}^s for any intermediate state $|k\rangle$

$$\Gamma_{i < j}^s = \Gamma_{i < k}^s + \Gamma_{k < j}^s. \quad (2.21)$$

The equations of motion for the atomic operators have the zero-curvature representation (2.17) independent of the constraint.

A zero-curvature representation implies that the system is integrable. Thus the model (2.15) can now be solved by the Quantum Inverse Scattering Method.

References

- [1] V. I. Rupasov, JETP Lett. 36 (1982) 142.
- [2] A. LeClair, *QED for a Fibrillar Medium of Two-Level Atoms*, hep-th/9604100.